TABLE III. The second column lists the predicted counting rates for production of submuons for a time in which four counts were actually observed. Column 3 lists the half-lives necessary to reduce the predicted rates to the observed rate of 4 counts. The total amount of absorber in the telescope was 179.0 g cm^{-2} of lead. The ranges listed for 175, 150, and 100 m_e particles were calculated from Barkas.[11] The other ranges were taken from the dashed curve of Fig. 3.

Submuon mass (m_e)	Predicted number of counts/10^{17} incident electrons	Half-life of submuons to give 4 counts (10^{-10} sec)	Predicted range in lead (g cm^{-2})
	(a) Spin-$\frac{1}{2}$ submuons:		
175	70	22	170.7
150	1200	9.4	182.2
100	3980	5.2	204.0
50	8410	2.3	227
25	12 500	1.1	236
	(b) Spin-0 submuons:		
175	10	68	170.7
150	190	14	182.2
100	660	7.0	204.0
50	1690	3.0	227
25	2870	1.4	236

CONCLUSIONS

The results of this experiment rule out any but very short-lived singly charged particles in the mass range 5—175 m_e. This result, plus the theoretical results on the vacuum polarization described in the introduction make it unlikely that charged particles with rest mass between that of the electron and muon exist.

ACKNOWLEDGMENTS

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Range of Very High-Energy Nucleon-Nucleon Collisions*

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The range of the chain-of-pions interaction is calculated for very high-energy nucleon-nucleon collisions in an approximation which does not require a complete dynamical description of the process. It is assumed that the chain-of-pions interaction is a primary process whose amplitude is not derived from that for "low-energy" processes. The interaction is described by two parameters, the average minimum-momentum transfer Δ_0 and the average fireball mass m_0. Certain results can be expressed in terms of Δ_0 and m_0 alone and are generally valid for all "linked-peripheral" models. In particular, if Δ_0 and m_0 are constant, then the inelasticity is constant, the number of fireballs is proportional to $\ln(s/M^2)$, where $s^{1/2}$ is the total barycentric energy, and the multiplicity is also proportional to $\ln(s/M^2)$. The chain-of-pions interaction in which the nucleons remain unexcited, N-N final states, is expected to be the most important process for small Δ_0 because of the considerably larger phase space available for it compared to that for isobar production. Thus, N-N final states give rise to the longest range part of the interaction and are estimated to make a larger contribution to the cross section than states in which even the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon isobar is produced. An additional result is that the iterated dominant "low-energy" pion-exchange model gives a nucleon-nucleon cross section of at most several mb if only low values of the momentum transfer of one of the nucleons or isobars are allowed. With the approximations used, it is then possible to calculate the long-range part of the elastic diffraction scattering amplitude in the almost transparent, purely absorbing, optical approximation. We obtain the Regge behavior in the limit of a large number of fireballs. At incident nucleon laboratory energy $E_L=10^3M$, the amplitude has not yet reached the asymptotic limit. For $\Delta_0^2=5m_\pi^2$ and $m_0=2M$, one finds that the inelasticity is $\frac{1}{3}$, the number of "fireballs" is two, and the range is in close agreement with that given by the one-pole elastic Regge amplitude with $\alpha'=1/M^2$. Finally, it is found that the nucleons which emerge unexcited in the final state lie within a cone whose angular width decreases with energy at a rate such that the transverse momentum P_T also decreases and $P_T \propto (\ln s)^{-1/2}$. This behavior is correlated to the shrinking of the elastic diffraction peak but is apparently in disagreement with high-energy events.

I. INTRODUCTION

IN many high-energy nucleon-nucleon collisions it is observed that the final-state particles have very small transverse momenta and that the secondary particles, mainly pions, appear to be produced in one or more groups called "fireballs."[1,2] It seems reasonable

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[2] A recent review of the data is given by D. H. Perkins, in *Proceedings of the International Conference on Theoretical Aspects of Very High-Energy Phenomena* (CERN, Geneva, 1961), p. 99.

to assume that some "peripheral" mechanism links the groups that are produced and provides a damping factor in the momentum transferred between them. One possibility, for example, is that each link consists of an exchanged virtual pion, leading to a chain-of-pions process and the damping factor is the pion propagator or "pole."[3] There are other suggestions as to the nature of the links[4,5] but the over-all kinematical picture is the same.

Goebel[6] first emphasized that the single-pion-exchange interaction can be expected to yield a range of interaction greater than m_π^{-1} which is usually associated with the exchange of a particle of mass m_π. The increased range arises from the chain-of-pions interaction contained in the model at higher energies. His conclusion appears to be of a more general nature. Any "linked-peripheral" process can be expected to give rise to a range of interaction greater than that given by any one of its components.

The effect of the increased range in the chain-of-pions interaction is born out in the work of Amati *et al.*[7] They examine a model in which it is assumed that the chain-of-pions interaction is a dominant process and that it can be related to a dominant "low-energy" single-pion-exchange interaction through an iterative procedure. With this model they find that the elastic diffraction scattering amplitude is of the Regge type. As has been frequently discussed,[8] the Regge behavior of the elastic-scattering amplitude, along with the reasonable assumptions that the nucleon-nucleon total cross section is constant at very high energies and that the interaction is purely absorbtive, imply that the range and transparency of the nucleon-nucleon interactions are increasing slowly with energy. The rates of increase are such that the total cross section remains constant.

It has been conjectured that the Regge behavior of the elastic diffraction amplitude continues to very high energies.[8] Starting from this conjecture, we find that the "linked-peripheral" production process represents one of the simplest mechanisms which can account for the increased range of the interaction and which is consistent with the data available at present. The spin of the particles can be assumed to make a negligible contribution. Experimentally[2] the number of particles produced is proportional to $s^{1/4}$, where $s^{1/2}$ is the total energy in the barycentric system, and the particles are mainly pions (spin zero). The elastic diffraction data indicate that important interactions occur in states of angular momentum $\alpha(s \ln s)^{1/2}$.[9] Therefore, the actual spin of the particles makes a vanishingly small contribution to the total angular momentum of the system and cannot be the source of the increase in the range of interaction.

It is of interest to see whether the "linked-peripheral" production of particles can produce an interaction range which is consistent with that obtained from the vacuum Regge pole fit of the elastic N-N scattering data at the accelerator energies. The range of the interaction is sensitive to the details of the links and it is calculated here in first approximation for the chain-of-pions graph. From the point of view of field theory or "polology" this process is expected to give the longest range interaction because the pion has the lightest mass of the strongly interacting particles. However, the methods used here are based upon kinematic approximations which are generally valid for "linked-peripheral" models.

The iterated-dominant "low-energy" pion-exchange model appears to be too great a simplification of the nucleon-nucleon interaction to be used to obtain a reliable estimate of its range. The assumption that the one-pion-exchange process is dominant at every stage of the iterative procedure can lead to a large accumulated error. We know now that links other than exchanged virtual pions are of interest[4,5] and that there is also the possibility of a significant number of events without a fireball structure.[2] A calculation based upon the iterated dominant "low-energy" pion-exchange model is made in Sec. V which shows that if only low values of the square of the four momentum transfer of one of the nucleons or isobars are allowed, then this model gives a nucleon-nucleon cross section of at most several mb.

Even if the chain-of-pions interaction (or more generally any "linked-peripheral" mechanism) is not the dominant process, it may still make a dominant contribution to the longest range part of the nucleon-nucleon interaction. It is also possible that the chain-of-pions interaction at high energies cannot be related to "low-energy" interactions through iterations because, for example, the interference effects may vary with energy. Therefore, we treat the chain of pions as a primary interaction. As the amplitudes are no longer tied to low-energy processes and the dynamical details of the vertices are not known, we introduce two parameters, the average minimum-momentum transfer Δ_0 and the average fireball mass m_0. This is sufficient

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[5] S. C. Frautschi, Nuovo Cimento **28**, 409 (1963).

[6] C. Goebel, in *Proceedings of the Midwest Theoretical Conference (1961)*; and in *Proceedings of the International Conference on Theoretical Aspects of Very High-Energy Phenomena* (CERN, Geneva, 1961), p. 353.

[7] D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, Nuovo Cimento **22**, 569 (1961); and D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962).

[8] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1961) [translation: Soviet Phys.—JETP **14**, 478 (1962)]; C. Lovelace, Nuovo Cimento **25**, 730 (1962); G. F. Chew and S. Frautschi, Phys. Rev. Letters **7**, 394 (1961); and **8**, 41 (1962); and R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

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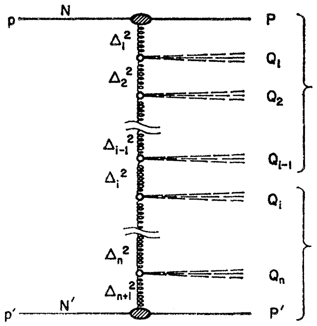

FIG. 1. A general "linked-peripheral" graph for the production of n fireballs. Nucleons N and N' are incident with four-momenta p and p', respectively. Two nucleons or isobars emerge, N or N^*, with four-momenta P and P', and n fireballs with four-momenta Q_i, $i=1,\cdots,n$. Each group is linked to the next by some peripheral mechanism which restricts the momentum transfer to small values. The momentum transfer for the ith link is indicated by Δ_i^2. The groups included in the brackets marked P_i and P_i' form the supergroups for the ith link, and $\Delta_i^2=-(p-P_i)^2$.

with certain other approximations to determine the interaction range.

In Secs. II, III, and IV we obtain general kinematical results for "linked-peripheral" interactions such as the inelasticity and the number of fireballs, which can be expressed in terms of the parameters Δ_0 and m_0 alone. Certain of these results are also given by Frautschi[5] for the model which he has suggested, but there are some important differences in the approximations used. In particular, we are interested in small Δ_0 and in the case that the nucleons are unexcited.

In Sec. V we determine which chain-of-pions graph is probably most important.

In Sec. VI the range is calculated in a first approximation in which it is assumed that the fireballs have a small fraction of the total energy in the barycentric system and that the vertex interactions do not have important spin or angular dependence. With this approximation it is possible to obtain the long-range part of the elastic diffraction scattering amplitude in the almost transparent purely absorbing optical approximation. We find the Regge behavior for the elastic scattering amplitude in the limit of a large number of fireballs.

II. NOTATION AND KINEMATICS

Nucleons N and N' are incident with four-momenta p and p', $p^2=p'^2=M^2$, where M is the nucleon mass and units with $\hbar=c=1$ are used. Two nucleons or nucleon isobars come off in the final state with four-momenta P and P', where $P^2=M_f^2$, $P'^2=M_f'^2$, and M_f and M_f' are the masses of the "isobars." There are n fireballs produced with momenta Q_i and masses m_i, $Q_i^2=m_i^2$ for $i=1, 2, \cdots, n$. The total momentum of the fireballs is given by $Q=\sum Q_i$ and the total mass is m, $Q^2=m^2$.

In a "linked-peripheral" production model, it is assumed that the momentum transfer associated with each link is small. This has the effect that the momentum of each of the "bodies" in the final state makes a small angle with the collision axis in the barycentric system. We assign each of the final state groups a position in a chain diagram, as shown in Fig. 1, according to the magnitude and sign of its three momentum in the barycentric system. The positive direction is chosen to be that of the projectile nucleon p, in the laboratory system.

The total barycentric energy $s^{1/2}$ is defined by

$$s=(p+p')^2. \tag{2.1}$$

The momentum transfer variable for the ith link, Δ_i^2, is given by

$$\Delta_i^2=-(p-P_i)^2, \quad \text{where} \quad P_i=P+\sum_{j=1}^{i-1} Q_j. \tag{2.2}$$

We also define

$$P_i'=P'+\sum_{j=i}^{n} Q_j. \tag{2.3}$$

In this notation $P_1=P$ and $P_{n+1}'=P'$. The groups composing P_i and P_i' are called the supergroups of the ith link. The mass of each supergroup $s_i^{1/2}$ is given by $s_i=P_i^2$.

The components of four-vectors in the barycentric system (B) are labeled with subscripts B and are given by: $p=(E_B,\mathbf{P}_B)$, $Q_i=(W_{iB},\mathbf{Q}_{iB})$, $Q=(W_B,\mathbf{Q}_B)$, $P_i=(s_{iB},\mathbf{P}_{iB})$ and similarly for the primed variables. The magnitudes of the three vectors are designated by $|\mathbf{p}_B|=p_B$, $|\mathbf{Q}_{iB}|=Q_{iB}$, etc.

For each supergroup P_i we define a scattering angle θ_i by the equation

$$\mathbf{p}_B\cdot\mathbf{P}_{iB}=p_BP_{iB}\cos\theta_i.$$

Equation (2.2) can be written as

$$(\Delta_i^2)=(\Delta_i^2)_{\min}+2p_BP_{iB}(1-\cos\theta_i), \tag{2.4}$$

where $(\Delta_i^2)_{\min}$ is a function of M^2, s_i, s_i', and s.

In general, if particles of mass m_1 and m_2 are incident with four momenta p_1 and p_2 and two groups with mass s_1 and s_2 emerge with four momenta P_1 and P_2, as shown in Fig. 2, and $\Delta^2=-(p_1-P_1)^2$, then the equation for the phase space boundary is given by[10]

$$\begin{aligned}(s_1-m_1^2)(s_2-m_2^2)&\\ +(1/s)(s_1m_2^2-s_2m_1^2)(s_1-m_1^2-(s_2-m_2^2))&\\ =\Delta^2s\bigg[1-\frac{s_1+s_2+m_1^2+m_2^2+\Delta^2}{s}&\\ +\frac{(m_1^2-m_2^2)(s_1-s_2)}{s^2}\bigg],&\end{aligned} \tag{2.5}$$

[10] F. Salzman and G. Salzman, Phys. Rev. **125**, 1703 (1962).

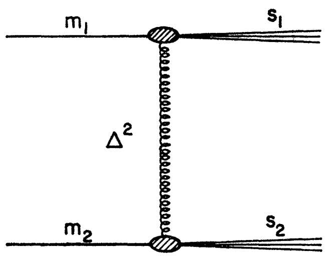

FIG. 2. A general "two-center peripheral" process. Particles m_1 and m_2 are incident with four-momenta p_1 and p_2, respectively, and two groups of particles emerge, s_1 and s_2, with four-momenta P_1 and P_2. The momentum transfer Δ^2 is given by $\Delta^2=-(p_1-P_1)^2$ and the energy variable $s=(p_1+p_2)^2$.

where $s=(p_1+p_2)^2$. This equation gives the minimum value of Δ^2 for given s, m_1^2, m_2^2, s_1, s_2 or alternatively, the maximum value of s_1 or s_2 for fixed values of the other parameters.

Equation (2.5) can be put into a more convenient form for our purposes by solving it in terms of the expression $s_1-m_1^2+\Delta^2$. It then becomes

$$s_1-m_1^2+\Delta^2=\frac{s}{2m_2^2}(s_2-m_2^2+\Delta^2)\left\{-\left(1-\frac{m_1^2+m_2^2}{s}\right)+\left(1-\frac{2(m_1^2+m_2^2)}{s}+\frac{(m_1^2-m_2^2)^2}{s^2}\right)^{1/2}\times\left(1+\frac{4m_2^2\Delta^2}{(s_2^2-m_2^2+\Delta^2)^2}\right)^{1/2}\right\}. \quad (2.6)$$

A useful approximation of Eq. (2.6) is

$$(s_1-m_1^2+\Delta^2)=(s-m_1^2-m_2^2)\left(\frac{s_2-m_2^2+\Delta^2}{2m_2^2}\right)\times\left\{-1+\left(1+\frac{4m_2^2\Delta^2}{(s_2-m_2^2+\Delta^2)^2}\right)^{1/2}\right\}$$

$$\text{for}\quad m_1^2+m_2^2\lesssim s/4. \quad (2.7)$$

We now let $m_1^2=m_2^2=M^2$. Equation (2.7) has two limiting cases of interest:

(1) $s_2=M^2$ (one nucleon emerges unexcited)

$$s_1-M^2+\Delta^2=s\frac{\Delta}{M}\left(1-\frac{\Delta}{2M}\right), \quad (2.8)$$

for $4M^2/s\ll1$ and $\Delta^2/4M^2\ll1$; and

(2) s_1, $s_2>M^2$ (excitation at each vertex)

$$s_1-M^2+\Delta^2=\frac{s\Delta^2}{s_2-M^2+\Delta^2},$$

$$\text{for}\quad 4M^2/s\ll1,\quad\text{and}\quad\frac{4M^2\Delta^2}{(s_2-M^2+\Delta^2)^2}\lesssim\frac{1}{2}. \quad (2.9)$$

As can be seen the approximate boundary equation for case (1) cannot be obtained from that for case (2) in the limit $s_2\rightarrow M^2$.

In order to make a comparison between (1) and (2), we let s_2 in Eq. (2.9) be equal to $M_{3/2}^2$, the mass of the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon resonance and the minimum isobar mass of interest. For small Δ/M, Eqs. (2.8) and (2.9) become

$$(1)\qquad s_1-M^2=s\Delta/M,$$

and (2.10)

$$(2)\qquad s_1-M^2=1.4s(\Delta/M)^2,\quad s_2=M_{3/2}^2.$$

Thus, there can be considerably larger values of s_1 for given Δ for (1), in which the nucleon emerges unexcited, than that for (2), in which just the lowest isobar is excited. Alternatively, for given s and s_1, the minimum values of Δ for the two cases, Δ_1 and Δ_2, are considerably different and $\Delta_1\ll\Delta_2$. This is due to the fact that the smallest values of $(\Delta^2)_{\min}$ occur for the least excitation at the two vertices.

III. APPROXIMATIONS AND INELASTICITY

The independent variables that are usually used to describe a process of the type shown in Fig. 1 are the two sets: (1) the energies of the supergroups $\{s_i\}$, and (2) the momentum transfers $\{\Delta_i^2\}$. However, we assume that the quantities $(\Delta_i^2)_{\min}$ defined in Eq. (2.4) are equal to some average value Δ_0^2,

$$(\Delta_i^2)_{\min}=\Delta_0^2\quad\text{for}\quad i=1,2\cdots,n+1,$$

and that the mass of each fireball is equal to some average mass m_0^2,

$$m_i^2=m_0^2\quad\text{for}\quad i=1,2\cdots,n.$$

Since the values of $(\Delta_i^2)_{\min}$ are fixed, we have in fact fixed the values of the set $\{s_i\}$. The energies s_i can be obtained by repeated application of the boundary equation given in Eq. (2.6) for specified M_f, M_f', s, Δ_0^2, and m_0^2.

It is also assumed that the total energy carried off by the fireballs in the barycentric system, W_B, is small compared to $s^{1/2}$ and that the momentum of each fireball is small compared to that of the nucleon "isobars." If we define the inelasticity I as

$$I=W_B/s^{1/2},$$

then experimentally[2] many events are observed with $I\lesssim\frac{1}{3}$.

For given final nucleon states, the inelasticity of a process is a function of Δ_0 alone. To see this, we note that

$$I=1-(E_B+E_B')/s^{1/2},$$

where E_B and E_B' are the barycentric energies of the nucleon isobars P and P', respectively, and are given by

$$E_B=\frac{s^{1/2}}{2}\left(1-\frac{s_1'-M_f^2}{s}\right),$$

and (3.1)

$$E_B'=\frac{s^{1/2}}{2}\left(1-\frac{s_{n+1}-M_f'^2}{s}\right).$$

In order to find the energies s_1' and s_{n+1}, it is helpful to look at the graph of Fig. 3 which shows in greater detail the parts of Fig. 1 which are of interest. The energies s_1' and s_{n+1} can be obtained by breaking the graph at the first and last link, respectively, comparing it with the general two-group graph of Fig. 2, and then making the proper substitutions in the general equation for the boundary given in Eq. (2.7). Because of the large difference in the phase space depending on whether nucleons N or isobars N^* are emitted, we examine the following three special cases: (1) final states in which both nucleons emerge unexcited, called N-N final states; (2) final states in which a nucleon and one isobar emerge, N-N^* final states; and (3) states in which two isobars emerge, N^*-N^* final states.

(1) *N-N* Final States

The energy s_1' can be obtained from the approximation given in Eq. (2.8) by making the substitutions $s_1\to s_1'$ and $\Delta\to\Delta_0$. Substituting this expression for s_1' into Eq. (3.1), we find

$$E_B=\frac{s^{1/2}}{2}\left(1-\frac{\Delta_0}{M}\left(1-\frac{\Delta_0}{2M}\right)\right),\quad \text{for } (s_1'-M^2)\gg\Delta_0^2. \quad (3.2)$$

It follows from the symmetry of the phase space, i.e., $(\Delta_i^2)_{\min}=\Delta_0^2$ and $M_f=M_f'=M$, that $E_B'=E_B$. Therefore, the total energy of the fireballs in the barycentric system is given by

$$W_B=s^{1/2}-2E_B=s^{1/2}\frac{\Delta_0}{M}\left(1-\frac{\Delta_0}{2M}\right). \quad (3.3)$$

In order to find the total mass of the fireball system m, we see that the s_1' part of the graph of Fig. 3 itself can be related to the general graph of Fig. 2 by the following substitutions:

$$\Delta^2\to\Delta_0^2;\quad m_1^2\to-\Delta_0^2;\quad s_2,\, m_2^2\to M^2;$$
$$s\to s_1';\quad \text{and}\quad s_1\to m^2.$$

Making these substitutions in the approximation for the boundary given by Eq. (2.8), we find

$$m^2+2\Delta_0^2=(s_1'-M^2+\Delta_0^2)\frac{\Delta_0}{M}\left(1-\frac{\Delta_0}{2M}\right), \quad (3.4)$$

provided $|\Delta_0^2-M^2|\lesssim s_1'/4$, $(\Delta_0^2+M^2)^2\ll s_1'^2$, and $\Delta_0^2\ll 4M^2$. The expression for s_1' can be obtained as before

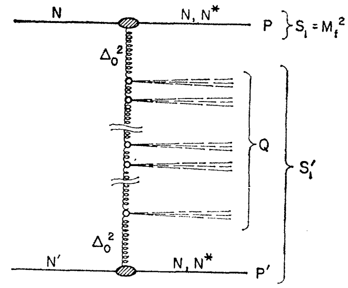

FIG. 3. The same process as shown in Fig. 1; however, the n fireballs are here treated as one group with four-momentum Q and rest mass m, $m^2=Q^2$.

from Eq. (2.8) with the substitutions $s_1\to s_1'$ and $\Delta\to\Delta_0$. When this result is substituted into Eq. (3.4), one finds

$$m^2+2\Delta_0^2=s\left(\frac{\Delta_0}{M}\right)^2\left(1-\frac{\Delta_0}{2M}\right)^2. \quad (3.5)$$

For $\Delta_0^2\ll m^2$, we see by comparing Eqs. (3.3) and (3.5) that $W_B^2\approx m^2$. Thus, for this symmetrical case, the total mass of the fireballs is at rest in the barycentric system.

(2) *N-N** Final States

Using a procedure analogous to that used in the previous case, we find

$$m^2+2\Delta_0^2=s\frac{\Delta_0}{M}\left(1-\frac{\Delta_0}{2M}\right)\left(\frac{M_f^2-M^2+\Delta_0^2}{2M^2}\right)\times\left\{-1+\left(1+\frac{4M^2\Delta_0^2}{(M_f^2-M^2+\Delta_0^2)^2}\right)^{1/2}\right\}. \quad (3.6)$$

To find W_B, we note that the energy E_B is given in Eq. (3.2) and the energy E_B' can be obtained by substituting Eq. (2.7) (with $s_2=M_f^2$) for s_1 into Eq. (3.1). We then find for E_B'

$$E_B'=\frac{s^{1/2}}{2}\left(1-\left(\frac{(M_f^2-M^2+\Delta_0^2)}{2M^2}\right)\times\left[-1+\left(1+\frac{4M^2\Delta_0^2}{(M_f^2-M^2+\Delta_0^2)^2}\right)^{1/2}\right]\right), \quad (3.7)$$

where we have assumed $s\gg M^2$, Δ^2 and $s_1\gg M_f^2$. This gives for $W_B=s^{1/2}-(E_B+E_B')$,

$$W_B=\frac{s^{1/2}}{2}\left\{\frac{\Delta_0}{M}\left(1-\frac{\Delta_0}{2M}\right)+\left(\frac{M_f^2-M^2+\Delta_0^2}{2M^2}\right)\times\left[-1+\left(1+\frac{4M^2\Delta_0^2}{(M_f^2-M^2+\Delta_0^2)^2}\right)^{1/2}\right]\right\}. \quad (3.8)$$

For the nonsymmetrical case, $M_f\neq M_f'$, we find that $W_B\neq m$ and the total fireball mass is not at rest in the barycentric system.

(3) N^*-N^* Final States

We shall assume that $M_f=M_f'$, so that we again have a symmetric case. Then $E_B=E_B'$ and is given by Eq. (3.7). We find for W_B,

$$W_B=s^{1/2}\frac{(M_f^2-M^2+\Delta_0^2)}{2M^2}\times\left[-1+\left(1+\frac{4M^2\Delta_0^2}{(M_f^2-M^2+\Delta_0^2)^2}\right)^{1/2}\right]. \quad (3.9)$$

We obtain for the fireball mass m,

$$m^2+2\Delta_0^2=s\left(\frac{M_f^2-M^2+\Delta_0^2}{2M^2}\right)^2\times\left[-1+\left(1+\frac{4M^2\Delta_0^2}{(M_f^2-M^2+\Delta_0^2)^2}\right)^{1/2}\right]^2, \quad (3.10)$$

and for $\Delta_0^2\ll m^2$ we find again that $W_B=m$.

The inelasticity, $I=W_B/s^{1/2}$ can be obtained for the three cases directly from Eqs. (3.3), (3.8), and (3.9), and is seen to depend only upon Δ_0 and the masses of the isobar states involved. Thus, if Δ_0 is constant with energy, then I is constant also.

In Table I we list the value of I for the three cases

TABLE I. Inelasticity.

Process (final states)	I: $\Delta_0^2=M^2/9$	I: $\Delta_0^2=M^2$
$N-N$	0.28	0.50
$N-N_{3/2}$	0.20	0.48
$N_{3/2}-N_{3/2}$	0.12	0.46

considered with $N^*=N_{3/2}$, the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon isobar and for two values of Δ_0^2, $\Delta_0^2=M^2/9$ and M^2. For $\Delta_0^2=M^2/9$, $I=0.28$ for N-N final states and drops to 0.12 for $N_{3/2}$-$N_{3/2}$ final states. For $\Delta_0^2=M^2$, $I\approx0.5$ for all three cases. It is interesting to note that it is possible to have inelasticities well in the physical range with values of $\Delta_0^2\ll M^2$ if the two nucleons emerge unexcited.

Small Δ_0^2 does not imply small Δ_i^2. It is still necessary to know the angular dependence of the supergroups in order to estimate the average value of Δ_i^2. Of course, small Δ_0^2 allows for the possibility of small Δ_i^2. In any case, we see that the condition that I be small places a restriction on the parameter Δ_0 which for values $\Delta_0^2\ll M^2$ depends strongly upon whether the nucleon is excited or unexcited.

IV. NUMBER OF FIREBALLS

In order to calculate the range of a "linked-peripheral" interaction it is necessary to know the number, n, of fireballs emitted as a function of the variables s, M_f, M_f', Δ_0^2, and m_0^2. To do this, the pertinent part of the graph of Fig. 3, consisting of the n fireballs that are emitted and the two links by which they are connected to the nucleon "isobar" lines, is shown enlarged in Fig. 4. We see that we can reduce this graph to that of the general two-group system considered in Sec. II with the following identifications: The end links correspond to incident "particles" of mass$-\Delta_0^2$; the barycentric energy is set equal to m; and the fireballs are divided into two groups, the lower group consisting of only the nth fireball with mass m_0, and the upper group of the remaining $n-1$ fireballs with a total mass m_{n-1}. The numbering, which is unimportant, is the same as that given in Fig. 1. If we now compare Fig. 4 with the general graph shown in Fig. 2, we see that the approximate general boundary equation given in Eq. (2.7) can be used with the following substitutions:

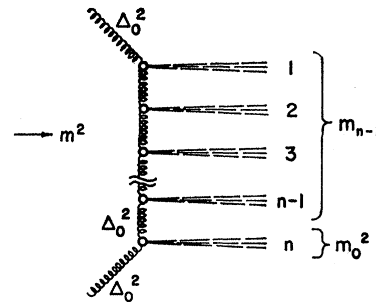

FIG. 4. The group of n fireballs of Figs. 1 and 3 shown in greater detail.

$$m_1^2,\, m_2^2\rightarrow -\Delta_0^2;\quad s\rightarrow m^2;\quad \Delta^2\rightarrow\Delta_0^2;$$
$$s_2\rightarrow m_0^2;\quad \text{and}\quad s_1\rightarrow m_{n-1}^2.$$

The resulting equation for m_{n-1}^2 is

$$m_{n-1}^2+2\Delta_0^2=\left(\frac{m^2+2\Delta_0^2}{m_0^2+2\Delta_0^2}\right)\Delta_0^2, \quad (4.1)$$

provided

$$4\Delta_0^2\lesssim m^2/2 \quad\text{and}\quad \frac{2\Delta_0^2}{m_0^2+2\Delta_0^2}\lesssim\frac{1}{2}.$$

If the number of fireballs is two, $n=2$, then the system m_{n-1} consists of one fireball. In Eq. (4.1) we set $m_{n-1}^2=m_0^2$ and $m^2=m_2^2$, where m_2 is the rest energy of the system of two fireballs, and find for m_2^2,

$$m_2^2+2\Delta_0^2=(m_0^2+2\Delta_0^2)^2/\Delta_0^2. \quad (4.2)$$

If $n=3$, the system m_{n-1} consists of two fireballs and in Eq. (4.1) we set $m_{n-1}^2=m_2^2$ and $m^2=m_3^2$. Substituting Eq. (4.2) for m_2^2 into Eq. (4.1), we find for m_3^2

$$m_3^2+2\Delta_0^2=(m_0^2+2\Delta_0^2)^3/(\Delta_0^2)^2. \quad (4.3)$$

Equation (4.3) is easily generalized to the case of n fireballs. If we set $m^2=m_n^2$, the rest energy of the system of n fireballs, we find

$$(m_n^2+2\Delta_0^2)=(m_0^2+2\Delta_0^2)^n/(\Delta_0^2)^{n-1}. \quad (4.4)$$

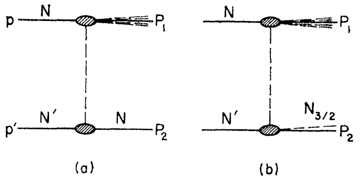

FIG. 5. Graphs for $N-N$ collisions in the single-pion-exchange model. The graphs differ only in that in (a) a nucleon N emerges from the lower vertex and in (b) the $\frac{3}{2}$–$\frac{3}{2}$ pion-nucleon isobar, $N_{3/2}$, emerges.

Solving Eq. (4.4) for n, we obtain

$$n=\frac{\ln[(m^2+2\Delta_0^2)/\Delta_0^2]}{\ln[(m_0^2+2\Delta_0^2)/\Delta_0^2]},$$

where the subscript n on m_n has been dropped. The final step is to relate m^2 to s. The relevant formula for N-N final states is given by Eq. (3.5), for N-N^* by Eq. (3.6) and for N^*-N^* by Eq. (3.10). For the case of interest, N-N final states, we have

$$n=\frac{\ln[(s/M^2)(1-\Delta_0/2M)^2]}{\ln[(m_0^2+2\Delta_0^2)/\Delta_0^2]}. \qquad (4.5)$$

In all three cases, we find that if m_0 and Δ_0 are constant with energy then $n \propto \ln(s/M^2)$.

V. DOMINANT CHAIN-OF-PIONS INTERACTION

We now consider specifically the "linked-peripheral" interaction consisting of a chain-of-pions. The part of the phase space given by the smallest values of Δ_i^2 for each link is expected to give rise to the longest range part of the interaction and to be the one in which the chain-of-pions graph is most likely the dominant process. In Sec. III it is shown that it is possible to obtain physically interesting values of the inelasticity for small Δ_0^2, $\Delta_0^2 \ll M^2$, if the nucleons emerge unexcited in the final state. However, the vertex interaction for isobar excitation, in particular that for the formation of the $N_{3/2}$ state, is stronger than the π-N-N vertex. It is, therefore, *a priori* possible that $N_{3/2}$ final states make a considerably larger contribution to the cross section and are of greater importance in determining the range of the interaction than the unexcited nucleon states.

In this section, we estimate the relative importance of final states in which N or $N_{3/2}$ emerge by evaluating the cross section for both graphs of Fig. 5 in the single-pion-exchange approximation. The two graphs of Fig. 5 differ only in that in (a) an N emerges and in (b) an $N_{3/2}$ emerges from the lower vertex. The upper vertex is taken to be the total π-N interaction. The notation is as follows: nucleons N and N' are incident with four-momenta p and p'; the group of particles which emerges from the upper vertex has four-momentum P_1 and rest energy $s_1^{1/2}$, where $s_1=P_1^2$; and the N or $N_{3/2}$ emerges with four-momentum P_2, where $P_2^2=M^2$ or s_2, respectively. Of course, the four-vectors have different components in the two cases because of the different masses.

The cross section for the diagram of Fig. 5(a) in which a nucleon N emerges and a pion of charge τ is exchanged, $d\sigma_N{}^\tau/d\Delta^2$, is given in the pole approximation by the Chew and Low formula[11]

$$\frac{d\sigma_N{}^\tau}{d\Delta^2}=\frac{4\pi f_\tau^2}{(2\pi)^2}\frac{M^2}{p^2 s}\frac{\Delta^2}{(\Delta^2+m_\pi^2)^2 m_\pi^2} \times \int p_{s_1} s_1 \sigma_{\pi^\tau-N}{}^{\mathrm{tot}}(s_1)\, d(s_1^{1/2}), \quad (5.1)$$

where

$$f_\tau^2=f_0^2\begin{Bmatrix}1 \text{ for } \pi^0 \text{ exchange}\\ 2 \text{ for } \pi^{+,-} \text{ exchange}\end{Bmatrix},$$

p_{s_1} is the magnitude of the initial momentum for pion-nucleon scattering at the barycentric energy $s_1^{1/2}$ and is given by

$$(p_{s_1}^2+M^2)^{1/2}=(s_1+M^2-m_\pi^2)/(2s_1^{1/2}), \qquad (5.2)$$

and $\sigma_{\pi^\tau N}{}^{\mathrm{tot}}(s_1)$ is the total π^τ-N cross section at energy s_1. The upper limit $(s_1)_{\max}$ of the integral for given Δ^2 is given by Eq. (2.8) and can be approximated by

$$(s_1)_{\max}=(s\Delta/M)(1-\Delta/2M), \qquad (5.3)$$

for $(s_1)_{\max} \gg M^2$, Δ^2. For sufficiently large s, the major contribution to the integral comes from large s_1 and we assume

$$\sigma_{\pi^\tau N}{}^{\mathrm{tot}}(s_1)=\sigma_{\pi N}{}^{\mathrm{tot}},$$

where $\sigma_{\pi N}{}^{\mathrm{tot}}$ is an average constant pion-nucleon cross section. For large s and s_1 we can also take $p=s^{1/2}/2$ and $p_{s_1}=s_1^{1/2}/2$. Making the indicated approximations, summing over the two-pion charge state (one neutral and one charged), and doing the s_1 integration with the use of Eq. (5.3) for the upper limit, we obtain

$$\sigma_N=\frac{3f^2}{2\pi}\sigma_{\pi N}{}^{\mathrm{tot}}\int^{(\Delta^2)_{\max}}\frac{\Delta^4(1-\Delta/2M)^2}{(\Delta^2+m_\pi^2)^2}\frac{d(\Delta^2)}{m_\pi^2}, \quad (5.4)$$

where $(\Delta^2)_{\max}$ is a cutoff needed to limit the Δ^2 integration to small values. The approximation for the upper limit of the s_1 integration, Eq. (5.3), is valid for $(\Delta^2)_{\max} \lesssim M^2$.

The cross section for the diagram of Fig. 5(b) is given by the formula for single-pion exchange in the two-center model in which the lower vertex group, P_2, is restricted to the $N_{3/2}$ state. For the exchange of a pion of charge τ, the cross section becomes, in the pole

[11] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

approximation,[10]

$$\frac{d\sigma_{N_{3/2}}{}^{\tau}}{d\Delta^2}=\frac{1}{4\pi^3p^2s}\frac{1}{(\Delta^2+m_\pi{}^2)^2}\int d(s_1{}^{1/2})p_{s_1}s_1\sigma_{\pi^\tau-N}{}^{\text{tot}}(s_1)\times\int d(s_2{}^{1/2})p_{s_2}s_2\sigma_{\pi^{-\tau}-N'}(s_2)\,, \quad (5.5)$$

where $\sigma_{\pi^{-\tau}-N'}(s_2)$ is the cross section for a pion of charge $-\tau$ incident upon nucleon N' at the barycentric energy $s_2{}^{1/2}$, and $s_2{}^{1/2}$ is restricted to lie in the energy range of the $\frac{3}{2}$-$\frac{3}{2}$ pion-nucleon resonance, $7.7m_\pi\leq s_2{}^{1/2}\leq 9.4m_\pi$. In order to simplify the integral we make an isobar approximation for the $N_{3/2}$ vertex by setting

$$\Lambda_{3/2}=\int_{7.7m_\pi}^{9.4m_\pi}d(s_2{}^{1/2})p_{s_2}s_2\sigma^{3/2}(s_2)\,, \quad (5.6)$$

where $\sigma^{3/2}$ is the cross section for the $J=\frac{3}{2}$, $T=\frac{3}{2}$ pion-nucleon state. We then set $P_2{}^2=M_{3/2}{}^2=1.72M^2$ in Eq. (2.7) for the upper limit of the s_1 integration. Summing over the three-charge states of the virtual pion, making the high-energy approximations used in the previous case, and doing the s_1 integration, we obtain

$$\sigma_{N_{3/2}}=\frac{1}{4\pi^3}\sigma_{\pi N}{}^{\text{tot}}\frac{\Lambda_{3/2}}{4M^4}\int^{(\Delta^2)_{\max}}d\Delta^2\frac{(0.72M^2+\Delta^2)^2}{(\Delta^2+m_\pi{}^2)^2}\times\left[-1+\left(1+\frac{4M^2\Delta^2}{(0.72M^2+\Delta^2)^2}\right)^{1/2}\right]^2, \quad (5.7)$$

where we have assumed $s\gg M^2$ and $s_1\gg M^2$, Δ^2. We have neglected the contribution to the Δ^2 integral from the lower s_1 limit.

For fixed $(\Delta^2)_{\max}$ both the N and $N_{3/2}$ cross sections in the pole approximation are constant in the high-energy limit. Direct evaluation of $\Lambda_{3/2}$ with $\sigma^{3/2}(s_2)$ taken to be the total π^+-p cross section yields $\Lambda_{3/2}=21.2M^2$. Using $f^2=0.08$ and taking $(\Delta^2)_{\max}\ll 0.72M^2$, we find from Eqs. (5.5) and (5.7) that $\sigma_{N_{3/2}}\approx(1/5)\sigma_N$. The small value of $\sigma_{N_{3/2}}$ compared to σ_N is due to the fact that the phase space for $N_{3/2}$ final states is considerably less than that for N because of the difference in mass. Evaluating σ_N and $\sigma_{N_{3/2}}$ for two values of $(\Delta^2)_{\max}$, i.e., $(\Delta^2)_{\max}=10m_\pi{}^2$ and $20m_\pi{}^2$, we find the results shown in Table II. We see that the total cross section for N final states is larger than that for $N_{3/2}$ for both values of $(\Delta^2)_{\max}$.

TABLE II. Cross sections for the graphs of Fig. 5.

$(\Delta^2)_{\max}$	(a) σ_N	(b) $\sigma_{N_{3/2}}$
$10m_\pi{}^2$	$0.16\sigma_{\pi N}{}^{\text{tot}}$	$0.025\sigma_{\pi N}{}^{\text{tot}}$
$20m_\pi{}^2$	$0.33\sigma_{\pi N}{}^{\text{tot}}$	$0.045\sigma_{\pi N}{}^{\text{tot}}$

It should be pointed out that because the equation for the phase space is so sensitive to the mass, the isobar approximation used in evaluating $\sigma_{N_{3/2}}$ is not too reliable and the values obtained should be viewed only as estimates. In addition, the pole approximation for $\sigma^{3/2}(s_2)$ neglects the important dependence on Δ^2 arising from the absorbtion of the virtual p-state meson and, thus, $\Lambda_{3/2}$ and $\sigma_{N_{3/2}}$ are underestimated. However, the result which we are primarily interested in is that N-final states can be expected to compete favorably with the $N_{3/2}$ states.

We have in effect evaluated the iterated dominant chain-of-pions model of Amati *et al.*, in the pole approximation. In their model, the upper π-N "vertex" of each graph in Fig. 5, which we have treated phenomenologically, consists entirely of chain-of-pions interactions. By taking the total π-N cross section to be ≈ 20 mb for the upper vertex, we are calculating an upper bound to the cross section coming from all chain-of-pion graphs in which an N or an $N_{3/2}$ is emitted at the lower vertex as shown in Fig. 5. We find the upper bound is ≈ 4 mb for momentum transfers at the N and $N_{3/2}$ vertices $\lesssim 10m_\pi{}^2$. If the π-N interaction is not predominantly chain-of-pions interactions, then the actual contribution from chain-of-pions graphs to the N-N cross section is considerably less.

In the model of Amati *et al.*, graphs in which an N or $N_{3/2}$ are emitted at a given vertex should give practically the whole N-N inelastic cross section which at very high energies is estimated to be ≈ 40 mb. This result can be shown in the following manner. One considers in addition all graphs in which the higher nucleon isobars are emitted from the fixed vertex. By a direct evaluation, the same as that done for the N and $N_{3/2}$ cases, it can be shown that these graphs are unimportant.[12] If now the excitation energy at the fixed vertex increases further, then one comes to the energies at which the π-N interaction is presumably dominated by the one-pion-exchange process and these graphs are already included in those for the N and $N_{3/2}$ cases.

The cross section for the chain-of-pions graphs obtained in the iterated approximation is very small. As mentioned in the Introduction we are adopting the point of view that the chain of pions is not a dominant process except in the longest range part of the interaction. In addition, we are considering it as a primary interaction which may not be simply related to lower energy interactions.

[12] F. Salzman (unpublished). One can proceed in this way and include a "core" part of the π-N interaction at the lower vertex, which can be "peripheral" but not reducible to a single-pion-exchange interaction. This, then, covers all the "peripheral" interaction in which at least *one link* is given by a single-pion-exchange process and avoids the overcounting problem contained in the two-center model approach. One finds that there must be a substantial contribution from the "core" part of the interaction if the single-pion-exchange "two-center" model is to explain a dominant part of the very high-energy N-N cross section.

VI. RANGE OF CHAIN-OF-PIONS INTERACTION

To obtain the range of the chain-of-pions interaction we refer again to the graph of Fig. 1 in which now all the links are assumed to be virtually exchanged pions. We take $(\Delta_i^2)_{\min}=\Delta_0^2$ for $i=1, 2, \cdots, n+1$ and $Q_i^2=m_0^2$ for $i=1, 2, \cdots, n$. As we have seen, for given s, Δ_0^2, and m_0^2 the supergroup energies s_i and s_i' are fixed. The remaining independent variables are Δ_i^2. Each Δ_i^2 can be expressed in terms of Δ_0^2 and the angle of the vector $\mathbf{P}_i$ with respect to $\mathbf{p}$ in the rest system of the vector $\mathbf{P}_{i+1}$ for $i=1, 2, \cdots, n+1$, where the vector $\mathbf{P}_{n+2}$ is given by $\mathbf{P}_{n+2}=\mathbf{P}+\mathbf{Q}+\mathbf{P}'=\mathbf{p}+\mathbf{p}'$. However, for the limiting case $\mathbf{Q}_{iB}\ll\mathbf{P}_B$, the barycentric system (B) is particularly simple to use.

The inelastic amplitude, f_C^{in}, is proportional to the product P of the $n+1$ pion propagators

$$f_C^{\text{in}}\propto P=\prod_{i=1}^{n+1}\frac{1}{m_\pi^2+\Delta_i^2}, \tag{6.1}$$

where all the spin dependence is neglected. Substituting for Δ_i^2 the expression given in Eq. (2.4), we obtain

$$P=\prod_{i=1}^{n+1}1/(2p_BP_{iB})(1/(1+\tfrac{1}{2}\eta_i^2-\cos\theta_i)),$$

where

$$\eta_i^2=(m_\pi^2+\Delta_0^2)/(p_BP_{iB}), \tag{6.2}$$

and we have set $(\Delta_i^2)_{\min}=\Delta_0^2$. Equation (6.2) is rewritten as

$$P=A\exp\left\{-\sum_i\ln\left(1+\frac{1-\cos\theta_i}{\frac{1}{2}\eta_i^2}\right)\right\},$$

where (6.3)

$$A=\left(\prod_i\frac{1}{2p_BP_{iB}}\right)\exp\{-\textstyle\sum_i\ln(\eta_i^2/2)\}.$$

We now assume that $\mathbf{Q}_{iB}\ll\mathbf{P}_B$ and $\mathbf{Q}_B\ll\mathbf{P}_B$ so that

$$\mathbf{P}_{iB}=\mathbf{P}_B+\sum_{j=1}^{i-1}\mathbf{Q}_{jB}\approx\mathbf{P}_B,$$

and (6.4)

$$\eta_i^2=\eta^2=(m_\pi^2+\Delta_0^2)/p_BP_B.$$

For small θ_i, the expression $1-\cos\theta_i$ in the exponent of Eq. (6.3) is $\approx\theta_i^2/2$ and can be related to the transverse momentum of the ith supergroup, given by $P_{iB}\sin\theta_i\approx P_{iB}\theta_i$. If we let φ_j be the angle between $\mathbf{Q}_{jB}$ and $\mathbf{p}_B$ and θ_N the angle between $\mathbf{P}_B$ and $\mathbf{p}_B$, then the transverse momentum of each supergroup for small θ_N and φ_j is given by

$$P_{iB}\theta_i\approx P_B\theta_N+\sum_{j=1}^{n-1}Q_{jB}\varphi_j,$$

where the transverse momentum of the nucleon is $P_B\theta_N$ and that of the jth fireball is $Q_{jB}\varphi_j$. For $Q_{jB}\varphi_j$ sufficiently small, the transverse momentum of each supergroup can be approximated by that due to the nucleon,

$$P_{iB}\theta_i\approx P_B\theta_N.$$

This approximation is not inconsistent with the experimental observation that the secondary pions have an average transverse momentum $\sim$0.4–0.5 BeV/c.[2] The transverse momenta of the fireball particles can be all due to their own relative motion in the fireball rest system and the total transverse momentum of each fireball itself is negligible in the barycentric system (B). Finally, we also make the approximation

$$\theta_i=\theta_N.$$

Inserting these approximations into Eq. (6.3), we obtain

$$f_C^{\text{in}}\propto P$$
$$=A\exp\{-(n+1)\ln[1+(1-\cos\theta_N)/(\eta^2/2)]\}. \tag{6.5}$$

If the vertex interactions do not depend significantly on the variables Δ_i^2 for Δ_i^2 small, then the main angular dependence of the amplitude f_C^{in} for small θ_N is given by Eq. (6.5). The range is closely related to the exponential falloff of the amplitude. For $n+1\gtrsim3$, the magnitude of the exponent will be $\lesssim1$ for $(1-\cos\theta_N)/(\eta^2/2)\lesssim1/(n+1)$. Since this is the range of values of interest, we expand the logarithm and obtain

$$f_C^{\text{in}}\propto A\exp\{-[2(n+1)/\eta^2](1-\cos\theta_N)\}. \tag{6.6}$$

The amplitude f_C^{in} depends only on the nucleon variable in the approximation that the fireballs have small total and negligible transverse momenta in the barycentric system. The problem then corresponds formally to that of a "two-body" inelastic state. If the Regge behavior of the elastic amplitude continues to very high energies and the total cross section remains constant, then the nucleon-nucleon interaction can be treated in the almost transparent purely absorbing optical approximation. In this approximation, if the amplitude for a dominant "two-body" inelastic channel is given by that of Eq. (6.6), then the elastic diffraction amplitude, f_C^{el}, which is required by the unitarity relation, is[13]

$$f_C^{\text{el}}\propto\exp[-((n+1)/\eta^2)(1-\cos\theta)], \tag{6.7}$$

where θ is the elastic scattering angle in the barycentric system.

The N-N elastic amplitude in the dominant vacuum Regge pole hypothesis is given by

$$f_R^{\text{el}}\propto\exp[-(2p_B^2\alpha'\ln s/s_0)(1-\cos\theta)], \tag{6.8}$$

where $s_0\approx2M^2$, the trajectory $\alpha(t)$ has been approximated by $\alpha(t)=1+t\alpha'$ and $t=-2p_B^2(1-\cos\theta)$ for elastic scattering. A fit of this expression to the N-N elastic data in the 3–30-BeV incident-nucleon laboratory energy range gives $\alpha'\approx1/M^2$.

[13] F. Salzman (to be published).

If $n \gg 1$, and Δ_0 and m_0 do not vary significantly with energy, then $\eta^{-2} \propto p_B^2$ and $n \propto \ln(s/M^2)$. In this case, the elastic diffraction scattering amplitude, f_C^{el}, obtained from the chain-of-pions process is of the Regge form given by Eq. (6.8) in the limit of large n.

The amplitudes of Eqs. (6.7) and (6.8) are of the form

$$f \propto \exp[-\beta(1-\cos\theta)].$$

The range R which corresponds to this amplitude is given by[13]

$$R = (2\beta)^{1/2}/p_B.$$

Using this expression, we find the range R_R of the amplitude f_R^{el} of Eq. (6.8) to be

$$R_R = (2/M)[\ln(s/2M^2)]^{1/2}, \qquad (6.9)$$

where we have taken $\alpha' = 1/M^2$, and the range R_C of the amplitude f_C^{el} of Eq. (6.7) to be

$$R_C = \left\{\frac{2(n+1)[1-(\Delta_0/M)(1-\Delta_0/2M)]}{(m_\pi^2+\Delta_0^2)}\right\}^{1/2},$$

where

$$n = \frac{\ln[(s/M^2)(1-\Delta_0/2M)^2]}{\ln[(m_0^2+2\Delta_0^2)/(\Delta_0^2)]}. \qquad (6.10)$$

In obtaining (6.10) we have substituted for η^2 the expression given in Eq. (6.4) and we have set $P_B = p_B[1-(\Delta_0/M)(1-\Delta_0/2M)]$ which is obtained from Eq. (3.2) for $s^{1/2} \gg M$.

A comparison of R_C and R_R can be made for given values of s, Δ_0^2, and m_0^2. The quantity Δ_0^2 is not known experimentally; but, with $\Delta_0^2 = 5m_\pi^2$, we see from Table I that for N-N final states the inelasticity $I = 0.3$ which is within the range of values observed experimentally. This value of I is somewhat high in terms of the approximations used which are satisfied for $I \ll 1$. Nevertheless, since all the data are lumped together, it is more interesting at this time to see what results one obtains with a value of I that is representative of a large number of the reported events. The incident-nucleon laboratory energy E_L is taken to be $\gtrsim 10^3 M$ so that it is reasonable to use the almost transparent, purely absorbing, optical approximation for the nucleon-nucleon interaction.

In Table III, R_C and R_R are compared for $E_L = 10^3 M$, $\Delta_0^2 = 5m_\pi^2$, and for three values of m_0^2: $m_0^2 = m_\rho^2 = 2/3M^2$, where m_ρ is the mass of the ρ meson (the lowest mass of interest), $2M^2$, and $4M^2$. Also shown, are the number, n, of fireballs produced as given by Eq. (4.5). Experimentally, at $E_L \approx 10^3 M$, the average fireball mass is $\approx 2M$ and the average number produced is ≈ 2. In all three cases the range R_C is larger than R_R. For the fireball mass $m_0^2 = 4M^2$, we find $n = 2$ and also the best fit to R_R.

TABLE III. Ranges R_C and R_R. The chain-of-pions range R_C is calculated with $\Delta_0^2 = 5m_\pi^2$.

	$E_L = 10^3 M$			$n, s/M^2 \gg 1$
m_0^2	n	R_C	$R_R = 5.3/M$	R_C
$2/3M^2$	3.7	$6.7/M$		$(2.2/M)\ln(s/M^2)^{1/2}$
$2M^2$	2.5	$5.8/M$		$(1.8/M)\ln(s/M^2)^{1/2}$
$4M^2$	2.1	$5.5/M$		$(1.6/M)\ln(s/M^2)^{1/2}$

The number of fireballs in all three cases considered in Table III is small which means that at this energy the corresponding elastic amplitudes have not yet reached the asymptotic Regge form. For this reason we include in Table III the asymptotic expression for the range obtained in the limit s/M^2, $n \gg 1$. The best limiting value in comparison to the expression for R_R given in Eq. (6.9) is that for an average fireball mass $m_0 = M$. Of course, all three values given in Table III are fairly close and the slope of the vacuum trajectory α' is subject to a fair amount of uncertainty, so that it is perhaps not worthwhile making a detailed quantitative comparison. However, it appears most reasonable to conclude that the two ranges are compatible.

Finally, it is of interest to consider the transverse momentum P_T of the nucleons as given in this picture. [The transverse momenta of the fireballs cannot be calculated because they have been taken to be negligibly small.] From the inelastic amplitude of Eq. (6.6) we see that the important values of θ_N are given by

$$\theta_N \lesssim \eta/(n+1)^{1/2},$$

so that

$$P_T \approx P_B \theta_N \lesssim [(1-\Delta_0/M)(m_\pi^2+\Delta_0^2)/(n+1)]^{1/2}, \qquad (6.11)$$

where we have substituted for η the expression given in Eq. (6.4) and we have used the approximation that $P_B = p_B(1-\Delta_0/M)$. For $\Delta_0^2 = 5m_\pi^2$ and $n = 2$ we find

$$P_T \lesssim 0.16 \text{ BeV}/c.$$

The nucleon transverse momentum is much smaller than the average, which is ~ 0.4 BeV/c. Experimentally, there is evidence that the heavy particles have transverse momenta ~ 1–2 BeV/c.

Equation (6.11) shows that the transverse momentum P_T of the nucleon goes to zero as $(1/n)^{1/2}$ or $[\ln(s/M^2)]^{-1/2}$ for large n and s. If this is also true for each fireball, then the approximations leading to Eq. (6.5) may still be reasonable.

VII. DISCUSSION AND CONCLUSION

The results which can be expressed in terms of Δ_0^2 and m_0^2 alone and are generally valid for any "linked-peripheral" production process are

(1) The inelasticity depends only on Δ_0^2, and is given for N-N, N-N^*, and N^*-N^* final states by Eqs. (3.3), (3.8), and (3.9), respectively.

(2) The number of fireballs depends upon s, Δ_0^2, and m_0^2 and is given for the specific case of N-N final states by Eq. (4.5).

(3) If Δ_0^2 and m_0^2 are constant with energy, then
(i) the inelasticity is constant;
(ii) the number, n, of fireballs increases with energy as $\ln(s/M^2)$; and,
(iii) the multiplicity is simply proportional to the number of fireballs and is also $\propto \ln(s/M^2)$.

Any "linked-peripheral" mechanism in which the nucleons emerge unexcited, with $\Delta_0^2=5m_\pi^2$ and $m_0^2=4M^2$, which is the most interesting case examined in the last section, gives an inelasticity $I=0.3$ and the number of fireballs $n=2$ for incident nucleon laboratory energy $E_L=10^3M$. Thus, just the assumption of a "linked-peripheral" production model leads naturally to an explanation of some of the main features of very high-energy N-N inelastic events.

We have shown, in addition, that the chain-of-pions interaction with values of the parameters which are in agreement with the inelastic data is not inconsistent with the Regge vacuum pole hypothesis for high-energy elastic scattering.

It should be pointed out that the approximations used may give rise to a large accumulated error because the fireball variables are neglected in a sum of terms in the exponent of the expression for the amplitude. This is offset to some extent by the fact that the number, n, of fireballs depends only logarithmically on s, Δ_0^2, and m_0^2.

We have seen that the inelastic amplitude as given by Eq. (6.6) depends exponentially on the nucleon variable, $f_C{}^{\text{in}} \propto \exp\{[-2(n+1)/\eta^2](1-\cos\theta_N)\}$ which with the approximations made is of the form needed to give the Regge-type behavior of the amplitude in the elastic channel. However, this predicts that the important values of the angle, θ_N, at which the unexcited nucleon emerges decrease as $[s\ln(s/M^2)]^{-1/2}$ for large n and s. This in turn implies that the nucleon transverse momentum P_T also decreases, $P_T\propto[\ln(s/M^2)]^{-1/2}$.[14] This apparently does not agree with the present cosmic-ray data. Although the analysis made here does not apply to the accelerator energies, it seems reasonable to expect qualitatively the same type of behavior at these energies if the source of the Regge behavior of the elastic amplitude is the same. In any case, it seems worthwhile to look for such a correlation between the angular dependence of the nucleons produced in the most peripheral inelastic events and the angular dependence of the elastically scattered nucleons. If, in fact, this effect is not present then we may conclude that the experimental region lies outside of the range of the model considered here.

Finally, we have not exhausted the chain-of-pions possibilities. It is still interesting to examine N-N^* and N^*-N^* production, particularly if the experiments show that larger values of Δ_0^2 are important. If the links are exponential factors, as suggested by Frautschi,[5] then the same approximations lead to an inelastic amplitude which has an exponential dependence upon the nucleon variable. It would be of interest to determine the range for these various possibilities, particularly if they are successful in explaining the inelastic events. At present, this approach appears to be a simple test of whether the dominant vacuum Regge pole hypothesis for the elastic N-N scattering interaction is in fact compatible with the high-energy inelastic data that is now available.

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[14] *Note added in proof.* L. Van Hove has shown in a recent paper that if he assumes certain simple forms for the inelastic final states, then in order to account for both the conjectured shrinking of the p-p elastic diffraction scattering at very high energies and the observed constancy of the average transverse momentum of the secondaries produced in the inelastic collisions, it is necessary to assume a certain amount of correlation between the secondaries. A very simple mechanism which Van Hove discusses is apparently illustrated by the model considered in this paper. The mechanism consists of the secondaries arising from first-generation primaries which are uncorrelated but which have an average transverse momentum which approaches zero at least as fast as R^{-1} $[(\ln(s/M^2))^{-1/2}$ in the case of the Regge pole conjecture$]$ as $s^{1/2}\rightarrow\infty$. The explanation of the constant average transverse momentum of the secondaries is then the same as that given in the above text.